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EQUIVOLUMINAL, LAMÉ-TYPE WAVES IN COMPOSITE HOLLOW CYLINDERS†

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Abstract—A description is given of axially symmetric, equivoluminal waves propagating along the axis of a composite cylinder made up of concentric hollow cylinders. The waves can be viewed as a superposition of SV-type waves reflected at the free boundaries, and involve motion in planes through the cylinder axis. Waves of this type can exist only if the shear wave velocity is the same in the various media of the composite cylinder and, in addition, the thicknesses of the different media are proportional to the zeros of a simple transcendental equation.

Harmonic elastic waves propagating along a bounded elastic continuum generally involve both dilatational and shear deformation. This is basically due to the fact that incidence of either a shear or a dilatational wave on a traction-free boundary generally produces both types of reflected waves[1]. There exist, however, certain cases of pure shear motion. Examples are the face-shear waves in a plate, and their circularly polarized counterpart in circular cylinders, i.e. the torsional waves. In these waves, the shear displacements are parallel to the free surfaces and perpendicular to the direction of propagation. This constitutes SH-wave type motion[1], and it is well known that SH-waves do not induce a reflected dilatational wave. Another interesting case of pure shear, or equivoluminal motion has been discussed by Lamé[2] for plates, and by the author[3] for hollow cylinders. In these Lamé-type waves, the shear displacements lie in the sagittal plane, i.e. the plane defined by the direction of propagation and the normal to the free surfaces. These waves may be viewed as a superposition of SV-type waves undergoing total reflection at the free boundaries.

In this paper, we investigate the existence of Lamé-type waves in a composite, hollow, circular cylinder made up of two concentric hollow cylinders adhering along a common cylindrical surface. The general case of three-dimensional waves in composite hollow cylinders has been treated by Armenakas[4]. However, although the Lamé-type waves are a particularly simple case of axisymmetric waves, the possibility of their existence has not been discussed to date. This is not surprising, because such waves generally do not exist in composite cylinders. They can exist, however, if the shear wave velocity in both media is exactly the same and the dimensions of the component cylinders satisfy certain relationships. Moreover, composite cylinders with more than two layers which can transmit Lamé-type waves may be similarly formed. In addition to their intrinsic value as solutions to the three-dimensional wave equations, such simple solutions are useful as convenient checkpoints for approximate theories.

The analytical treatment follows Ref. [3]:

Consider the cylinder whose cross section is shown in Fig. 1. Waves are assumed to propagate along the z-axis, which is normal to the plane r,θ. Hereafter, a subscript i will be used to identify the two different regions of the composite cylinder, with $i = 1$ identifying the inner, and $i = 2$ the outer region. According to Ref. [3], Lamé-type displacements are of the form

$$\begin{aligned} u_{ri} &= \xi[A_iJ_1(\xi r) + B_iY_1]\cos(\omega t + \xi z) \\ u_{zi} &= -\xi[A_iJ_0(\xi r) + B_iY_0(\xi r)]\sin(\omega t + \xi z) \\ u_{\theta i} &= 0, \qquad (i = 1, 2) \end{aligned} \tag{1}$$

where J_n and Y_n are the Bessel functions of order n and of the first and second kind,

†Preliminary results of this investigation were published as an IBM Research note, NC-679 (1966) under the same title.

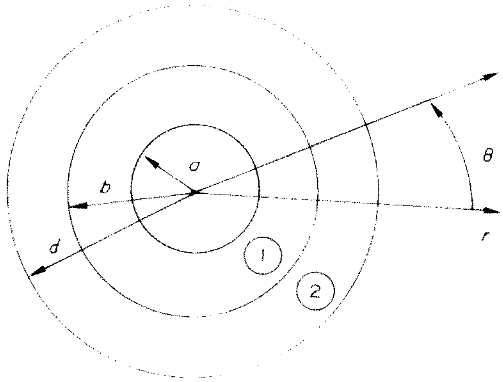

Fig. 1. Coordinates and dimensions.

respectively. The displacements given in eqn (1) satisfy the equations of motion provided that

$$2\xi^2 = \omega^2/V_s^2. \tag{2}$$

In eqn (2), V_s^2 is the square of the shear velocity, the same in both regions, given by

$$V_s^2 = \mu_1/\rho_1 = \mu_2/\rho_2 \tag{3}$$

where μ_i is the shear modulus and ρ_i the material density in region i. In addition, we must satisfy certain boundary conditions expressing the fact that the boundaries $r = a$ and $r = d$ are traction-free and that the displacements and traction are continuous at $r = b$.

The stresses σ_{rr}, σ_{rz}, and $\sigma_{r\theta}$, corresponding to the displacements (1), are[3]

$$\begin{aligned}(\sigma_{rr})_i &= -2\mu_i\xi^2[A_iJ_1'(\xi r) + B_iY_1'(\xi r)]\sin(\omega t + \xi z)\\ (\sigma_{rz})_i &= (\sigma_{r\theta})_i = 0\end{aligned} \tag{4}$$

where primes denote differentiation and respect ot r. Accordingly, the boundary conditions are

$$\begin{aligned}&A_1J_1'(\xi a) + B_1Y_1'(\xi a) = 0\\ &A_2J_1'(\xi d) + B_2Y_1'(\xi d) = 0\\ &(A_1 - A_2)J_1(\xi b) + (B_1 - B_2)Y_1(\xi b) = 0\\ &(A_1 - A_2)J_0(\xi b) + (B_1 - B_2)Y_0(\xi b) = 0\\ &(\mu_1A_1 - \mu_2A_2)J_1'(\xi b) + (\mu_1B_1 - \mu_2B_2)Y_2'(\xi b) = 0.\end{aligned} \tag{5}$$

In general, the five boundary conditions cannot all be satisfied simultaneously. However, there exist special values of the ratios a/b and a/d for which eqns (5) can be satisfied by an appropriate choice of the frequency ω. Setting

$$\begin{aligned}A_1 &= A_2\\ B_1 &= B_1\end{aligned} \tag{6}$$

we satisfy the third and fourth of eqns (5). The remaining ones become

$$\begin{aligned}&A_1J_1'(\xi a) + B_1Y_1'(\xi a) = 0\\ &A_1J_1'(\xi d) + B_1Y_1'(\xi d) = 0\\ &(\mu_1 - \mu_2)[A_1J_1'(\xi b) + B_1Y_1'(\xi b)] = 0\end{aligned} \tag{7}$$

It is clear that eqns (7) are satisfied if

$$\frac{J_1'(\xi a)}{Y_1'(\xi a)} = \frac{J_1'(\xi b)}{Y_1'(\xi b)} = \frac{J_1'(\xi d)}{Y_1'(\xi d)} = c \text{ (constant)}. \tag{8}$$

Hence, equivoluminal waves exist in composite cylinders with dimensions a, b and d corresponding to the solution of eqns (8) for some value of c. A graphical representation of the solution is shown in Fig. 2, where setting $c = -1$ we obtain, for example

$$\xi a = 2{\cdot}8177$$
$$\xi b = 6{\cdot}1392$$
$$\xi d = 12{\cdot}4962$$

corresponding to

$$\omega = 3{\cdot}9848\,(V_s/a)$$
$$a/b = 0{\cdot}4590$$
$$a/d = 0{\cdot}2255$$

For very thin cylinders, all three of the arguments ξa, ξb and ξd must be very large. We may then approximate the Bessel functions by the first term of the Hankel semiconvergent series; namely

$$\sqrt{\left(\frac{1}{2}\pi x\right)}\, J_1'(x) \approx \cos\,(x - (\pi/4))$$
$$\sqrt{\left(\frac{1}{2}\pi x\right)}\, Y_1'(x) \approx \sin\,(x - (\pi/4)) \tag{9}$$

hence

$$\frac{J_1'(s)}{Y_1'(x)} \approx \cot\,(x - (\pi/4)) \tag{10}$$

and

$$\xi a \approx (\pi/4) + \cot^{-1} c$$
$$\xi b \approx \xi a + n(\pi/2) \tag{11}$$
$$\xi d \approx \xi a + m(\pi/2)$$

where m and n are integers. Hence, in order for Lamé-type waves to exist, the ratio of the two thicknesses of the two component cylinders must be very close to a rational number. It must be exactly a rational number in the case of a composite plate which is a limiting case of a cylinder, as the ratio of composite thickness to mean radius tends to zero.

It may be observed that the set of the ratios of the roots of eqn (8), for any c is dense in the open inverval from zero to one. Hence, any thickness ratios are arbitrarily close to a set satisfying eqns (8). However, in most practical cases one is interested in low frequency modes,

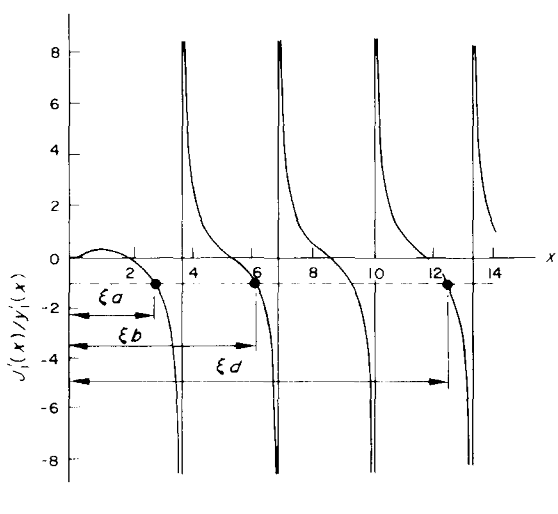

Fig. 2. Graphical determination of the ratios $a : b : d$.

and hence in cylinders with thickness ratios satisfying eqn (8), where ξa, ξb and ξd are selected from among the first few branches of the solution to the equation.

$$\frac{J_1'(x)}{Y_1'(x)} = c. \tag{12}$$

We may also observe that particularly simple solutions eqn (8) are obtained for $c = 0$, or for $c = \infty$. In the former case, the solution is

$$\begin{aligned} A_1 &= A_2 \neq 0 \\ B_1 &= B_2 = 0 \\ J_1'(\xi a) &= J_1'(\xi b) = J_1'(\xi d) = 0 \end{aligned} \tag{13}$$

and in the latter case

$$\begin{aligned} A_1 &= A_2 = 0 \\ B_1 &= B_2 \neq 0 \\ Y_1'(\xi a) &= Y_1'(\xi b) = Y_1'(\xi d) = 0 \end{aligned} \tag{14}$$

It is seen from the preceding discussion that the various material regions of the composite cylinder are in effect decoupled, and they vibrate in unison. In fact, satisfaction of the third of eqns (7) implies that all stresses at the interface between material regions are zero. It may also be observed that the discussion can be extended to the case of composite cylinders made up of more than two concentric cylinders. So long as the boundary radii are proportional to roots of eqn (12), there exists a Lamé-type mode analogous to that described for the two-layer cylinder.

Finally, if the shear moduli and densities of all the different regions are exactly identical, the cylinder acts as a homogeneous cylinder insofar as all pure shear modes are concerned, even though the Poisson's ratios of the different regions may be quite different. The results of Ref. [3] are applicable in this special case.

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